SPACELIKE SURFACES OF CONSTANT GAUSSIAN CURVATURE IN LORENTZ-MINKOWSKI SPACE

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Abstract. In this paper we survey several results on the study of spacelike surfaces of constant Gaussian curvature $K < 0$ in Lorentz-Minkowski space of dimension (2+1). Moreover, we show that the space of entire K-surfaces with bounded second fundamental form, up to translations, is naturally parameterized by an infinite-dimensional vector space, namely the tangent space at the trivial point of universal Teichmüller space.

1. Background and statement of the problem

Lorentz-Minkowski space is the vector space $\mathbb{R}^3$ endowed with a non-degenerate bilinear form of signature (2,1). Using coordinates $\mathbf{x} = (x_1, x_2, x_3)$ on $\mathbb{R}^3$, Lorentz-Minkowski space can thus defined as

$$\mathbb{L}^3 = (\mathbb{R}^3, \langle \mathbf{x}, \mathbf{x} \rangle = x_1^2 + x_2^2 - x_3^2) \ .$$

Lorentz-Minkowski space, in the following simply Minkowski space, is the analogue in Lorentzian geometry of Euclidean space:

$$\mathbb{E}^3 = (\mathbb{R}^3, \mathbf{x} \cdot \mathbf{x} = x_1^2 + x_2^2 + x_3^2) \ .$$

The group of isometries of $\mathbb{L}^3$ is thus identified as:

$$\mathrm{Isom}(\mathbb{L}^3) \cong \mathrm{O}(2,1) \rtimes \mathbb{R}^3 \ ,$$

where $\mathrm{O}(2,1)$ is the group of linear isometries of the standard bilinear form of signature $(2,1)$, and $\mathbb{R}^3$ acts by translations. This is indeed the analogue of the isomorphism

$$\mathrm{Isom}(\mathbb{E}^3) \cong \mathrm{O}(3) \rtimes \mathbb{R}^3 \ .$$

It is well known that there exists an isometric embedding $\iota : \mathbb{H}^2 \hookrightarrow \mathbb{L}^3$ of the hyperbolic plane $\mathbb{H}^2$ into Minkowski space. The hyperbolic plane is the unique (up to isometries) simply-connected, complete Riemannian surface of constant curvature -1. In this paper, we will mostly use the Klein model of $\mathbb{H}^2$, namely:

$$\mathbb{H}^2 = \left(\mathbb{D}, \frac{|dz|^2}{1 - |z|^2} + \frac{(z \cdot dz)^2}{(1 - |z|^2)^2} \right) \ , \tag{1}$$

where $\mathbb{D} = (z \in \mathbb{C} : |z| < 1)$ denotes the unit disc.

The image of the embedding ι is one sheet of the two-sheeted hyperboloid:

$$\iota(\mathbb{H}^2) = \{ \mathbf{x} \in \mathbb{L}^3 \, : \, \langle \mathbf{x}, \mathbf{x} \rangle = -1, x_3 > 0 \} \ .$$

The embedding ι is the analogue in Minkowski space of the isometric embedding of the round sphere $\mathbb{S}^2$ into Euclidean space. It turns out that the standard embedding $\mathbb{S}^2 \hookrightarrow \mathbb{E}^3$ is the unique isometric embedding of $\mathbb{S}^2$, up to post-composition with *global* isometries of $\mathbb{E}^3$.

A remarkable difference of Minkowski geometry with Euclidean geometry is the existence of *non-standard embeddings* of $\mathbb{H}^2$ into $\mathbb{L}^3$, that is, whose image is different from an isometric copy of the hyperboloid $\iota(\mathbb{H}^2)$. The first examples were constructed by Hano and Nomizu in [8].

1.1. **Formulation of the problem.** Hence a natural problem to consider is the following:

Problem 1.1. *Classify all isometric (or more generally, homothetic) embeddings of the hyperbolic plane into $\mathbb{L}^3$, up to post-composition with orientation-preserving isometries of $\mathbb{L}^3$.*

However, Problem 1.1 is still open at the time of writing the present paper. Observe that, if a spacelike surface Σ is the image of an homothetic embedding of $\mathbb{H}^2$ into $\mathbb{L}^3$, then the induced metric on Σ is a complete metric of constant Gaussian curvature $K < 0$. (If the embedding is isometric, then $K = -1$, namely we have a complete hyperbolic metric.)

Remark 1.1. *A basic remark is that the completeness of the induced metric on a spacelike surface Σ implies that the surface is an* entire graph*, namely*

$$\Sigma = graph(f) = \{(x_1, x_2, f(x_1, x_2)) : (x_1, x_2) \in \mathbb{R}^2\} ,$$

for some function $f : \mathbb{R}^2 \to \mathbb{R}$. This follows from the fact that the vertical projection $(x_1, x_2, x_3) \mapsto (x_1, x_2)$ is distance-increasing as a map from Σ to $\mathbb{R}^2$. In particular, under this hypothesis Σ is necessarily simply-connected. Since Σ is spacelike, the function f satisfies the condition $||\mathrm{grad} f|| < 1$.

Recall that the hyperbolic plane $\mathbb{H}^2$ is, up to isometries, the unique simply-connected, complete Riemannian surface of constant curvature -1. Thus surfaces Σ which are the images of a homothetic embedding of $\mathbb{H}^2$ can thus be characterized as spacelike entire graphs whose induced Riemannian metric is of constant curvature $K < 0$ *and* complete.

The main difficulty in approaching Problem 1.1 consists in dealing with the condition that the induced metric on Σ is a *complete* hyperbolic metric. Hence we will actually consider a more general problem, namely:

Problem 1.2. *For fixed $K < 0$, classify all convex spacelike surfaces Σ in $\mathbb{L}^3$ of constant Gaussian curvature K, which are entire graphs, up to orientation-preserving isometries of $\mathbb{L}^3$.*

For simplicity, we call a surface Σ *entire* if $\Sigma = graph(f)$ for some function $f : \mathbb{R}^2 \to \mathbb{R}$. Since we are interested in spacelike surfaces, we will always assume $||\mathrm{grad} f|| < 1$. Let us make some comments on this second version of our problem.

- As explained above, the Problem 1.2 is more general than Problem 1.1. In fact, an entire surface Σ of constant curvature $K < 0$ in $\mathbb{L}^3$ is not necessarily complete. Hence a K-surface Σ might be only *locally* homothetic to $\mathbb{H}^2$, but not *globally*.
- The condition of being an entire graph does not depend on the choice of a spacelike plane of $\mathbb{L}^3$. That is, if Σ is a graph over the plane

$x_3 = 0$, then it is a graph over *any* spacelike plane. Roughly speaking, the condition of being an entire graph on the one hand avoids the presence of singularities of lightlike type, as we shall see later. On the other hand, by this assumption, a subset of an entire graph is not considered as a different solution, and thus entire K-surfaces satisfy a kind of "maximality" condition.

- By the Lorentzian version of Gauss' equation,
$$K = -\det B \ ,$$
where B is the shape operator of Σ, regardless of the choice of future unit normal vector field on Σ. Convexity of Σ is thus implied by the condition $K < 0$. Hence we are considering smooth (strictly) convex spacelike surfaces such that the product of the principal curvatures is constant.
- Up to composing with the (time-reversing, orientation-preserving) isometry $(x_1, x_2, x_3) \mapsto (x_1, -x_2, -x_3)$, we can restrict ourselves to surfaces Σ which are *future-convex*, namely, such that the side of Σ pointing in the direction of $(0, 0, 1)$ is convex.
- In conclusion, we will actually consider future-convex entire K-surfaces, up to *time-preserving*, orientation-preserving isometries of $\mathbb{L}^3$. That is, up to the action of the group
$$\mathrm{Isom}_0(\mathbb{L}^3) \cong \mathrm{SO}_0(2,1) \rtimes \mathbb{R}^3 \ ,$$
where the 0 subscript denotes the connected component of the identity.

1.2. **Asymptotic behavior.** Heuristically, the reason of the remarkable difference between the rigidity of the isometric embeddings $\mathbb{S}^2 \hookrightarrow \mathbb{E}^3$, and the flexibility of the isometric embeddings $\mathbb{H}^2 \hookrightarrow \mathbb{L}^3$, is the non-compactness of $\mathbb{H}^2$. As a consequence, a statement of *existence and uniqueness* for entire K-surfaces Σ in $\mathbb{L}^3$ will be obtained once the *asymptotic behavior* of Σ is prescribed. The asymptotic behavior is encoded in a function on the circle, defined in the following way (see [3, 15]):

Definition 1.1. *Given a convex spacelike entire surface $\Sigma = graph(f)$ for $f : \mathbb{R}^2 \to \mathbb{R}$, we define the function $\varphi_\Sigma : \mathbb{S}^1 \to \mathbb{R} \cup \{+\infty\}$ as:*
$$\varphi_\Sigma(e^{i\theta}) = \lim_{r \to +\infty} (r - f(re^{i\theta})) \ .$$

Another way to describe the function φ_Σ is the following (see [1, 4]). Observe that every lightlike plane of $\mathbb{L}^3$ has the form
$$P = \{\mathbf{x} \in \mathbb{L}^3 \ : \ \langle \mathbf{x}, (e^{i\theta}, 1) \rangle = a\}$$
for some $e^{i\theta} \in \mathbb{S}^1 \subset \mathbb{R}^2$ and $a \in \mathbb{R}$. Now, it turns out (see [1, Section 2.3]) that for every $e^{i\theta} \in \mathbb{S}^1$, the lightlike plane
$$P = \{\mathbf{x} \in \mathbb{L}^3 \ : \ \langle \mathbf{x}, (e^{i\theta}, 1) \rangle = \varphi_\Sigma(e^{i\theta})\}$$
is a support plane of Σ. Namely, every vertical translate of P in the future direction intersects Σ, while every translate in the past vertical direction is disjoint from Σ. This point of view will be relevant below, when translating our problem in terms of the *support function*.

Conversely, given a function $\varphi : \mathbb{S}^1 \to \mathbb{R} \cup \{+\infty\}$, one can construct a convex domain, which will be called a *deformation of the cone*, to which any convex entire spacelike surface Σ satisfying $\varphi_\Sigma = \varphi$ will be asymptotic. Recall that the future cone over the origin in $\mathbb{L}^3$ is:

$$\mathrm{C}(0) := \{\mathbf{x} \in \mathbb{L}^3 \,:\, \langle \mathbf{x}, \mathbf{x} \rangle \leq 0, x_3 > 0\} \ , \tag{2}$$

and the lightlike support planes of $\mathrm{C}(0)$ are precisely the planes of the form

$$P = \{\mathbf{x} \in \mathbb{L}^3 \,:\, \langle \mathbf{x}, (e^{i\theta}, 1) \rangle = 0\}$$

for any $e^{i\theta} \in \mathbb{S}^1$.

Definition 1.2. *A* deformation of the cone *is a convex domain of the form:*

$$\mathrm{C}(\varphi) = \bigcap_{e^{i\theta} \in \mathbb{S}^1} \left\{ \mathbf{x} \in \mathbb{L}^3 \,:\, \langle \mathbf{x}, (e^{i\theta}, 1) \rangle \leq \varphi(e^{i\theta}) \right\} \ ,$$

for some function from $\varphi : \mathbb{S}^1 \to \mathbb{R} \cup \{+\infty\}$.

It turns out that φ can always be supposed lower-semicontinuous, as we will explain in the next section. If Σ is a convex spacelike entire surface such that $\varphi_\Sigma = \varphi$, then Σ is asymptotic to $\mathrm{C}(\varphi)$, in the sense that Σ and $\mathrm{C}(\varphi)$ have the same lightlike support planes. In the language of Lorentzian geometry, Σ is a *Cauchy surface* for $\mathrm{C}(\varphi)$, and $\mathrm{C}(\varphi)$ is the *domain of dependence* of Σ. If φ takes finite values on at least three distinct point, then the deformation $\mathrm{C}(\varphi)$ is called a *regular domain*, see also [2].

1.3. **Three important examples.** Let us now show some examples of the above constructions, which are actually very important in the proofs of several existence theorems in the literature, as we will explain in Section 3. We shall start from the most basic example, namely the future cone.

Example 1.1. *If $\varphi \equiv 0$ is the constant null function, then the associated convex domain $\mathrm{C}(0)$ is the future cone itself, as in Equation (2). The standard hyperboloid $\iota(\mathbb{H}^2)$ is asymptotic to $\mathrm{C}(0)$.*

Let us now consider a less trivial example.

Example 1.2. *Consider the function*

$$\varphi^h_{a,b}(e^{i\theta}) = \begin{cases} a\cos\theta & \text{if } \cos\theta \geq 0 \\ b\cos\theta & \text{if } \cos\theta \leq 0 \end{cases} \ , \tag{3}$$

for some fixed numbers $a \leq b$. It turns out that the domain $\mathrm{C}(\varphi^h_{a,b})$ is the future *of the segment $\overline{\mathbf{pq}}$ connecting the points $\mathbf{p} = (a, 0, 0)$ and $\mathbf{q} = (b, 0, 0)$. Namely, $\mathrm{C}(\varphi^h_{a,b})$ is the union of the translates of the future cone having basepoints on the segment $\overline{\mathbf{pq}}$. See Figure 1.*

A relevant observation is that the domain $\mathrm{C}(\varphi^h_{a,b})$ in Example 1.2 is invariant by a 1-parameter subgroup of $\mathrm{SO}_0(2,1)$, namely the subgroup

$$\mathcal{H} := \left\{ H_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix} \,:\, t \in \mathbb{R} \right\} \ .$$

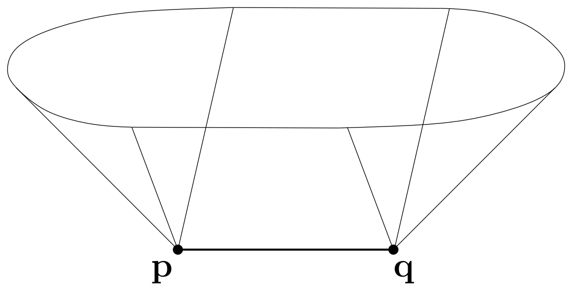

FIGURE 1. The domain $\mathrm{C}(\varphi^h_{a,b})$.

Hence, a simple way to produce a spacelike surface Σ asymptotic to $\mathrm{C}(\varphi^h_{a,b})$ is to take the $\mathcal{H}$-orbit of a spacelike curve contained in the (x_1, x_3)-plane, asymptotic to the lines $x_3 = -x_1 + a$ and $x_3 = x_1 - b$. That is, consider

$$\Sigma_g = \{H_t \cdot (s, 0, g(s)) \,:\, s \in \mathbb{R}\}\ ,$$

where g satisfies $|g'(s)| < 1$ for every s, and

$$\lim_{s\to+\infty}(s - g(s)) = b \qquad \lim_{s\to-\infty}(g(s) - s) = a\ .$$

It is essentially in this way that Hano and Nomizu produced the first examples of non-standard embeddings of $\mathbb{H}^2$ into $\mathbb{L}^3$, by expressing the condition of having constant Gaussian curvature in terms of an ordinary differential equation on the function g, and proving that the corresponding solutions provide surfaces Σ_g with complete induced metric. See [8].

The third example is somehow similar to Example 1.2, with the difference that the invariance by a hyperbolic subgroup is replaced by a parabolic subgroup.

Example 1.3. *Consider the function*

$$\varphi^p_c(e^{i\theta}) = \begin{cases} -c & \text{if } \theta = 0 \\ 0 & \text{otherwise} \end{cases}\ . \tag{4}$$

for a fixed constant $c > 0$. This means that the lightlike support planes of $\mathrm{C}(\varphi^p_c)$ coincide with the support planes of the future cone $\mathrm{C}(0)$, except for one plane which is translated vertically. The domain $\mathrm{C}(\varphi^p_c)$ is thus the future *of the parabola $\mathrm{C}(0) \cap P$, which is obtained as intersection of $\mathrm{C}(0)$ with the plane*

$$P = \{\mathbf{x} \in \mathbb{L}^3 \,:\, x_3 = x_1 + c\}\ .$$

Namely, $\mathrm{C}(\varphi^p_c)$ is the union of all translates of $\mathrm{C}(0)$ having basepoints on the parabola $\mathrm{C}(0) \cap P$.

As already mentioned, the domains in Example 1.3 are now invariant for a 1-parameter subgroup $\mathcal{P}$ of $\mathrm{SO}_0(2,1)$ of parabolic type, that is, $\mathcal{P} = \{P_t \,:\, t \in \mathbb{R}\}$ where P_t are linear parabolic isometries fixing the vector $(1, 0, 1)$. Similarly to Example 1.2, a way to produce convex spacelike entire surface asymptotic to $\mathrm{C}(\varphi^p_c)$ is to take the $\mathcal{P}$-orbit of a curve of the form $\{(s, 0, g(s)) \,:\, s \in \mathbb{R}\}$, where $g : \mathbb{R} \to \mathbb{R}$ satisfies $|g'(s)| < 1$ and

$$\lim_{s\to+\infty}(g(s) - s) = c \qquad \lim_{s\to-\infty}(g(s) - s) = 0\ .$$

2. Monge-Ampère equations

The purpose of this section is to translate Problem 1.2 in more analytical terms, related to the theory of Monge-Ampère equations. The fact that questions concerning spacelike surfaces of constant Gaussian curvature are related to Monge-Ampère equations is evident on one hand from the analogy with Euclidean geometry, and on the other hand from the following equation:

$$\det D^2 f = \frac{|K|}{(1-(\partial_{x_1} f)^2 - (\partial_{x_2} f)^2)^2} \ , \tag{5}$$

which expresses the condition that a surface $\Sigma = graph(f)$ has constant Gaussian curvature $K < 0$.

2.1. **Support functions.** However, given the nature of the problem and of Equation (5), it turns out that it is more convenient to express the condition of having constant Gaussian curvature in terms of a function on the unit disc. This is called the *support function* and is the analogue of the classical support function for convex bodies in Euclidean space.

Definition 2.1. *Given a convex spacelike surface $\Sigma \subset \mathbb{L}^3$, the* support function *of Σ is the function*

$$U_\Sigma : \mathrm{C}(0) \to \mathbb{R} \cup \{+\infty\}$$

defined by

$$U_\Sigma(\mathbf{x}) = \sup_{\mathbf{p}\in\Sigma} \langle \mathbf{x}, \mathbf{p} \rangle \ . \tag{6}$$

It can be easily checked that U is 1-homogeneous, and it is thus determined by its restriction

$$u_\Sigma := U_\Sigma|_{\mathrm{C}(0)\cap\{x_3=1\}} \ . \tag{7}$$

The set of definition of u_Σ is the disc

$$\mathrm{C}(0) \cap \{x_3 = 1\} = \{(x_1, x_2, 1) \, : \, x_1^2 + x_2^2 < 1\} \ ,$$

which can actually be interpreted as the Klein model of hyperbolic plane. It will thus be identified to the unit disc $\mathbb{D} = \{|z| < 1\}$. It then turns out that u_Σ is convex and extends uniquely to a lower-semicontinuous function on $\overline{\mathbb{D}}$.

Remark 2.1. *If $\Sigma = graph(f)$ is an entire surface (hence with $||\mathrm{grad} f|| < 1$), then under the above identification, u_Σ coincides with the* Legendre transformation *f^* of f, on the subset of $\mathbb{D}$ where it takes finite values. In fact, applying Equation (6) to a point in $\mathbb{D}$ and using that points of Σ have the form $(w, f(w))$, one obtains:*

$$u_\Sigma(z) = \sup_{w\in\mathbb{R}^2} (z \cdot w - f(w)) = f^*(z) \ ,$$

provided z is in the domain of definition

$$\mathcal{D}_\Sigma := \left\{ z \in \mathbb{R}^2 \, : \, \sup_{w\in\mathbb{R}^2} (z \cdot w - f(w)) < +\infty \right\}$$

of f^*. It turns out that such domain of definition $\mathcal{D}_\Sigma$ is a convex subset of $\mathbb{D}$, and it coincides with the convex envelope of the points $e^{i\theta} \in \mathbb{S}^1$ such that $u_\Sigma(e^{i\theta}) < +\infty$.

We will now use the notion of support function (or Legendre transformation) in order to reformulate Problem 1.2 as a partial differential equation of Monge-Ampère type on (a subset of) the unit disc.

2.2. **Analytic formulation.** The following properties will be important in order to reformulate Problem 1.2. Given a convex spacelike surface Σ, let u_Σ be its support function as in Definition 2.1.

- The surface Σ has constant Gaussian curvature $K<0$ if and only if

$$\det D^2 u_\Sigma(z) = \frac{1}{|K|}\frac{1}{(1-|z|^2)^2} \ , \tag{A}$$

  for every $z \in \mathcal{D}_\Sigma$.
- The surface Σ is an entire graph if and only if

$$\mathrm{im}(\mathrm{grad} u_\Sigma) = \mathbb{R}^2 \ , \tag{B}$$

  that is, if the gradient map of u_Σ, as a map from $\mathcal{D}_\Sigma$ to $\mathbb{R}^2$, is surjective.
- Given a lower-semicontinuous function $\varphi : \mathbb{S}^1 \to \mathbb{R} \cup \{+\infty\}$, the surface Σ is asymptotic to the deformation of the cone $\mathrm{C}(\varphi)$ if and only if

$$u_\Sigma|_{\partial\mathbb{D}} = \varphi \ , \tag{C}$$

  where $\partial\mathbb{D}$ is identified to $\mathbb{S}^1$.

There are two remarks concerning the last point. First, as already mentioned, this last condition (C) is equivalent to the condition that the asymptotic function φ_Σ of Σ coincides with φ. (Recall Definition 1.1.) Second, the support function u_Σ turns out to be continuous and convex in $\mathcal{D}_\Sigma$, but in general it is only lower-semicontinuous up to $\partial\mathbb{D}$. Hence the condition (C) should be meant in the following sense: given any parameterized line segment $\alpha : [0,\epsilon) \to \overline{\mathbb{D}}$ with $\alpha(0) \in \partial\mathbb{D}$,

$$\lim_{t\to 0^+} u_\Sigma(\alpha(t)) = \varphi(\alpha(0)) \ .$$

Finally, let us remark that the surface Σ can be easily recovered from the support function u_Σ. For instance, since the Legendre transformation is an involution, Σ turns out to be the graph of the Legendre transformation of u_Σ. Hence we can summarize the above discussion in the following characterization:

Lemma 2.1. *A convex spacelike surface Σ is an entire K-surface asymptotic to the deformation of the cone $\mathrm{C}(\varphi)$, for $\varphi : \mathbb{S}^1 \to \mathbb{R}\cup\{+\infty\}$ lower-semicontinuous and $K<0$, if and only of its support function $u_\Sigma : \mathbb{D} \to \mathbb{R}$ satisfies (A),(B),(C). Conversely, any such solution u of (A),(B),(C) uniquely determines an entire spacelike K-surface asymptotic to $\mathrm{C}(\varphi)$.*

2.3. **State-of-the-art: entire surfaces.** In this subsection we list results on the existence and uniqueness of solutions $u : \mathbb{D} \to \mathbb{R}$ of (A),(B),(C), for a given function φ in a certain regularity class.

Theorem 2.1 ([10])**.** *For every $K<0$ and every $\varphi \in C^\infty(\mathbb{S}^1)$, there exists a unique solution $u : \mathbb{D} \to \mathbb{R}$ of (A),(B),(C).*

Theorem 2.1 was actually proved in [10] for any dimension. The following result is an improvement in dimension (2+1), for Lipschitz regularity of φ:

Theorem 2.2 ([7]). *For every $K < 0$ and every $\varphi \in C^{0,1}(\mathbb{S}^1)$, there exists a unique solution $u : \mathbb{D} \to \mathbb{R}$ of* (A),(B),(C).

Finally, the following is the most general result in dimension (2+1) to the knowledge of the author, at the time of writing this paper:

Theorem 2.3 ([1]). *For every $K < 0$ and every $\varphi : \mathbb{S}^1 \to \mathbb{R}$ bounded and lower-semicontinuous, there exists a unique solution $u : \mathbb{D} \to \mathbb{R}$ of* (A),(B),(C).

Let us remark that, as already observed, the function φ can always be taken lower-semicontinuous, hence the only real hypothesis of Theorem 2.3 is the boundedness of φ. We will now give a geometric interpretation to this condition, and for this purpose we will need the following lemma:

Lemma 2.2. *Let $\mathbf{x} \mapsto m \cdot \mathbf{x} + v$ be an isometry of $\mathbb{L}^3$, with $m \in \mathrm{O}(3)$. Given a convex spacelike surface Σ, let $\Sigma' = m \cdot \Sigma + \mathbf{v}$. Then*

$$U_{\Sigma'}(\mathbf{x}) = U_\Sigma(m^{-1} \cdot \mathbf{x}) + \langle \mathbf{v}, \mathbf{x} \rangle \ . \tag{8}$$

In particular,

$$u_{\Sigma'}(z) = u_\Sigma(m^{-1}(z)) + v_1 z_1 + v_2 z_2 - v_3 \ , \tag{9}$$

where $z \in \mathbb{D}$, $\mathbf{v} = (v_1, v_2, v_3)$, and $m(z)$ denotes the induced action of m by isometries on the Klein model $\mathbb{D}$.

The next lemma then follows straightforwardly from Equation (9), by applying a vertical translation of the form $(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 - c)$ (so that m is the identity matrix).

Lemma 2.3. *Let $\varphi : \mathbb{S}^1 \to \mathbb{R}$ be lower-semicontinuous. Then $\mathrm{C}(\varphi)$ is contained in a translate of $\mathrm{C}(0)$ if and only if φ is bounded.*

We say that a deformation of the cone $\mathrm{C}(\varphi)$ is *of bounded type* if it is contained in a translate of the future cone. Hence a restatement of Theorem 2.3 is the following:

Corollary 2.1. *Given any deformation of the cone $\mathrm{C}(\varphi)$ of bounded type, there exists a unique entire spacelike K-surface asymptotic to $\mathrm{C}(\varphi)$.*

We conclude this subsection by a formulation of the most general possible conjecture about existence and uniqueness of entire K-surfaces.

Conjecture 2.1. *For every $K < 0$ and every $\varphi : \mathbb{S}^1 \to \mathbb{R}$ finite on at least three points, there exists a unique solution $u : \mathbb{D} \to \mathbb{R}$ of* (A),(B),(C). *Namely, there exists a unique entire K-surface asymptotic to every regular domain.*

Recall that the definition of regular domain was provided at the end of Subsection 1.2. Indeed, one can show that there can be no entire K-surface Σ for which the asymptotic function φ_Σ is finite only on 0, 1 or 2 points. On the contrary, there are several examples of entire K-surfaces Σ such that φ_Σ is infinite on large subsets of $\mathbb{S}^1$.

2.4. **State-of-the-art: boundedness of principal curvatures.** Corollary 2.1 essentially gives a solution to Problem 1.2 — namely, the classification of future-convex entire K-surfaces — under the additional hypothesis that the surface is asymptotic to a deformation of the cone of bounded type. Conjecture 2.1 would lead to a complete solution of Problem 1.2. In fact, in Problem 1.2 we asked for an understanding of those surfaces *up to isometries*, but this equivalence relation is easily understood using Lemma 2.2. In Subsection 5.3, we will discuss the correct parameter space to classify surfaces of bounded type up to isometries.

The purpose of this subsection instead consists in discussing another variation of Problem 1.1, namely a classification of entire K-surfaces *with bounded principal curvatures*. Recall that the principal curvatures of a spacelike surface Σ are the eigenvalues of the shape operator, say λ_1 and λ_2. They are non-negative if Σ is future-convex, if the shape operator is computed by using the future-directed unit normal vector field. By the Gauss' equation in Minkowski space, if Σ is a K-surface (for $K < 0$), then

$$\lambda_1 \lambda_2 = |K| \ .$$

We say that the principal curvatures are *bounded* if there exists $C > 0$ such that

$$\frac{1}{C} < \lambda_1, \lambda_2 < C \ .$$

Remark 2.2. *Suppose Σ is a convex spacelike entire K-surface with bounded principal curvatures. Then it is not difficult to show that the Gauss map $G : \Sigma \to \mathbb{H}^2$ is bi-Lipschitz with respect to the induced metric on Σ, and therefore Σ is complete. Hence the induced metric of Σ is a complete metric of constant curvature $K < 0$ — namely, Σ is the image of a homothetic embedding of $\mathbb{H}^2$.*

This shows that the condition of having bounded principal curvatures, for an entire K-surface, is actually stronger than the condition considered in Problem 1.1.

Theorem 2.4 ([10])**.** *Let Σ be a spacelike entire K-surface, which is asymptotic to $\mathrm{C}(\varphi_\Sigma)$. The following implications hold:*

$$\varphi_\Sigma \in C^\infty(\partial\mathbb{D}) \Rightarrow \Sigma \text{ has bounded principal curvatures } \Rightarrow \varphi_\Sigma \in C^0(\partial\mathbb{D}) \ .$$

However, neither of the two implications is sharp. The following theorem provided a complete characterization of the condition of boundedness of principal curvatures:

Theorem 2.5 ([1])**.** *Let Σ be a spacelike entire K-surface, which is asymptotic to $\mathrm{C}(\varphi_\Sigma)$. Then Σ has bounded principal curvatures if and only φ_Σ has the Zygmund regularity.*

We say that a function $\varphi : \mathbb{S}^1 \to \mathbb{R}$ has the *Zygmund regularity* if there exists a constant $C > 0$ such that, for every $\theta, h \in \mathbb{R}$,

$$|\varphi(e^{i(\theta+h)}) + \varphi(e^{i(\theta-h)}) - 2\varphi(e^{i\theta})| < C|h| \ .$$

We observe that the Zygmund regularity is a regularity class strictly contained between Lipschitz continuity and α-Hölder continuity, for every $\alpha \in$

$(0,1)$. Hence the Zygmund class is a larger class than the classes initially considered, for the existence problem, in Theorems 2.1 and 2.2.

In Section 5 we will discuss precisely a parameterization of the space of entire K-surfaces with bounded principal curvatures, up to the natural action by isometries of $\mathbb{L}^3$, in terms of universal Teichmüller space.

3. Explicit barriers

In this section we will briefly discuss some geometric ideas which are essential to the proofs of Theorems 2.1, 2.2 and 2.3 — namely, the results of existence and uniqueness for entire K-surfaces.

3.1. **Sketch of the argument.** In all the three cases, the first step of course is the construction of a smooth solution u of Equation (A), satisfying the boundary condition (C). We will not enter into the details of the proof of existence and uniqueness of such solutions — let us just say that this applies some important results on the theory of Monge-Ampère equations, with the additional difficulty that the factor on the right-hand side of Equation (A) blows up at the boundary of $\mathbb{D}$. Nevertheless, the most tricky part in the proofs is to show that the constructed solution satisfies the condition (B). This ensures that the K-surface constructed from the solution u does not develop lightlike rays — or in other words, it does not touch the boundary of the deformation of the cone $\mathrm{C}(\varphi)$.

For this purpose, the use of *barriers* is of key importance. The idea is to prove *directly* the existence of a solution to (A),(B) with a special boundary condition $u|_{\partial\mathbb{D}} = \varphi_0$. Then, by applying the maximum principle, one can use this explicit solution (the *barrier*) to prove that the solution u in the general case also satisfies the property (B).

More precisely, the argument goes as follows. Suppose by contradiction the surface Σ, obtained from a solution u of Equation (A) with $u|_{\partial\mathbb{D}} = \varphi$, is not entire. This means that the closure of Σ intersects the boundary of the deformation of the cone $\mathrm{C}(\varphi)$. Let $e^{i\theta_0}$ be the point of $\mathbb{S}^1$ corresponding to the lightlike support plane of $\mathrm{C}(\varphi)$ at $\overline{\Sigma}\cap\partial\mathrm{C}(\varphi)$, i.e. such that the lightlike plane

$$P_0 = \{\mathbf{x}\in\mathbb{L}^3 \,:\, \langle\mathbf{x},(e^{i\theta_0},1)\rangle = \varphi(e^{i\theta_0})\}$$

intersects $\overline{\Sigma}$. We will then produce a function $\varphi_0 : \mathbb{S}^1 \to \mathbb{R}$ such that:

- $\varphi(e^{i\theta}) \leq \varphi_0(e^{i\theta})$ for every $e^{i\theta}\in\mathbb{S}^1$;
- $\varphi(e^{i\theta_0}) = \varphi_0(e^{i\theta_0})$;
- φ_0 is the asymptotic function of an entire spacelike K-surface Σ_0.

Once this is achieved, an application of the maximum principle for Monge-Ampère equations proves that the support functions of Σ and Σ_0 satisfy $u_\Sigma \leq u_{\Sigma_0}$. Namely, that the surface Σ is contained in the future of the surface Σ_0. But Σ_0 is entire, hence disjoint from the plane P_0. This will contradict the fact that the closure of Σ intersects P_0.

3.2. **Construction of the barriers.** In order to produce a function φ_0 as above, one needs to produce explicit barriers Σ_0 *by hands* in some special cases, and to use the hypothesis on the regularity of φ. More concretely, in the cases of 2.1, 2.2 and 2.3 respectively:

(1) For Theorem 2.1, one can use as a barrier (translates of) the hyperboloid of Example 1.1. From Lemma 2.2, since the asymptotic function of the standard hyperboloid is the zero constant function, one sees that the asymptotic functions φ_{Σ_0} of translates Σ_0 of the hyperboloid are restrictions to $\mathbb{S}^1$ of affine functions on $\mathbb{R}^2$ (i.e. $\varphi_0(e^{i\theta}) = a\cos\theta + b\sin\theta + c$ for some $a, b, c \in \mathbb{R}$). Hence if φ is smooth, one can use such translates as barriers in correspondence of every point, by choosing a, b, c properly.

(2) For Theorem 2.2, one uses barriers which are invariant for a 1-parameter hyperbolic group of isometries, as in Example 1.2. In fact, as already explained, the condition of being a K-surface reduces to an ordinary differential equation, and an analysis of the solutions shows the existence of entire K-surfaces with asymptotic function $\varphi^h_{a,b}$ as in Equation 3. Up to choosing the parameters a and b of $\varphi^h_{a,b}$ suitably, and composing with isometries of $\mathbb{L}^3$, one produces barriers Σ_0 whose asymptotic function φ_0 is continuous and affine on two closed intervals I_1 and I_2, with $I_1 \cup I_2 = \mathbb{S}^1$ (where the slope of φ_0 on I_1 and I_2 can be made arbitrarily large). Hence this construction can be applied under the hypothesis of Lipschitz regularity of φ in the boundary condition (C).

(3) For Theorem 2.3, the necessary barriers arise (up to composing with isometries) from the situation of Example 1.3. In fact, the function φ^p_c introduced in Equation (4) is a "jump" function. Using the parabolic 1-parameter invariance, Equation (A) is again reduced to an ordinary differential equation, and it is possible to prove the existence of solutions whose $\mathcal{P}$-orbit are entire K-surfaces with asymptotic function φ^p_c. Now, composing this with isometries of $\mathbb{L}^3$, one can obtain entire K-surfaces Σ_0 with asymptotic function $\varphi_0 = \varphi_{\Sigma_0}$ an arbitrary lower-semicontinuous jump function. That is, $\varphi_0(e^{i\theta_0}) = a$ and $\varphi_0(e^{i\theta}) = b$ for every $\theta \neq \theta_0$, where θ_0 and $a < b$ can be chosen arbitrarily. Therefore the parabolic-invariant entire K-surfaces are well suited to be used in the argument above, when the only hypothesis on φ are boundedness and lower semicontinuity.

4. Hyperbolic plane geometry

In this section we will sketch some arguments in the proof of Theorem 2.5. The proof deeply involves tools from Monge-Ampère equations and from hyperbolic plane geometry, relying on the natural vector space isomorphism between $\mathbb{L}^3$ and the Lie algebra $\mathfrak{so}(2,1)$ (which is peculiar of dimension (2+1)).

4.1. Measured geodesic laminations.

To prove the double implication of Theorem 2.5, we prove that both conditions are equivalent to the finiteness of a certain quantity arising in hyperbolic geometry, namely the *Thurston norm* of a *measured geodesic lamination*. Let us see more precisely what this means.

Given a function $\varphi : \mathbb{S}^1 \to \mathbb{R} \cup \{+\infty\}$, let us consider the boundary $\partial C(\varphi)$ of the convex domain $C(\varphi)$. Although this is not a smooth surface, one can still consider its support function as in Definition 2.1 and Equation (7). Let us denote it by $v_\varphi : \mathbb{D} \to \mathbb{R} \cup \{+\infty\}$. It turns out that v_φ coincides with the convex envelope of φ, and thus it satisfies the equation $\det D^2 v_\varphi = 0$, in the weak sense of Monge-Ampère measures. This implies that v_φ is roughly speaking "piecewise affine": more precisely, the graph of v_φ is a *pleated surface* in $\mathbb{D} \times \mathbb{R}$. Recall that, in the Klein model of the hyperbolic plane $\mathbb{H}^2$ (see (1)), geodesics are straight lines in $\mathbb{D}$, and therefore the space of geodesics of $\mathbb{H}^2$ is:

$$\mathcal{G}(\mathbb{H}^2) = (\mathbb{S}^1 \times \mathbb{S}^1 \setminus \mathrm{diag}(\mathbb{S}^1))/\sim ,$$

where $\sim$ is the equivalence relation $(a, b) \sim (b, a)$. Hence v_φ determines a *bending lamination*, which is the data of:

- A *geodesic lamination* of $\mathbb{H}^2$, i.e. a closed subset of $\mathcal{G}(\mathbb{H}^2)$, made of disjoint geodesics (which encodes the locus of pleating);
- A *locally finite Borel measure* on $\mathcal{G}(\mathbb{H}^2)$ (which encodes the intensity of pleating), supported on the geodesic lamination of the previous point.

We are now ready to introduce the important definition of *bounded* measured geodesic lamination:

Definition 4.1. *Given a measured geodesic lamination μ, the* Thurston norm *of μ (which can equal $+\infty$ in general) is defined as:*

$$||\mu||_{\mathrm{Th}} := \sup_I \mu(\mathcal{G}_I) \in \mathbb{R} \cup \{+\infty\} ,$$

where $\mathcal{G}_I$ denotes the subset of $\mathcal{G}(\mathbb{H}^2)$ which intersects a geodesic segment I of $\mathbb{H}^2$, and I varies over all geodesic segments of length 1 transverse to the geodesic lamination. We say that μ is bounded *if $||\mu||_{\mathrm{Th}} < +\infty$.*

The proof of Theorem 2.5 then consists in showing that both conditions (the condition that φ has the Zygmund regularity, and that the entire K-surface asymptotic to $\mathrm{C}(\varphi)$ has bounded principal curvatures) are equivalent to $||\mu_\varphi||_{\mathrm{Th}} < +\infty$, where μ_φ is the bending lamination of v_φ.

4.2. **Infinitesimal earthquakes.** For the first equivalence mentioned above, we need to introduce yet another notion, namely the notion of *left (or right) earthquake map*. Intuitively, this is a (non-continuous, in general) map $E^l_\mu : \mathbb{H}^2 \to \mathbb{H}^2$ which is an isometry on any stratum of a measured geodesic lamination of $\mathbb{H}^2$ (the source), and the behavior of E^l_μ on two different strata differs by a hyperbolic isometry, which slides on the left (resp. on the right) accoding to the mass of the measured geodesic lamination on the subset of geodesics in $\mathcal{G}(\mathbb{H}^2)$ which separate the two strata. See [14, 9] for the precise definition. See also Figure 2.

The importance of earthquake maps was highlighted by Thurston, by means of the following theorem:

Theorem 4.1 ([14, 6, 12, 11])**.** *Given any orientation-preserving homeomorphism $h : \mathbb{S}^1 \to \mathbb{S}^1$, there exists a unique left earthquake map $E^l_\mu : \mathbb{H}^2 \to \mathbb{H}^2$, for a measured geodesic lamination μ, which extends to h on the boundary of $\mathbb{D}$. Moreover, h is quasisymmetric if and only if μ is bounded.*

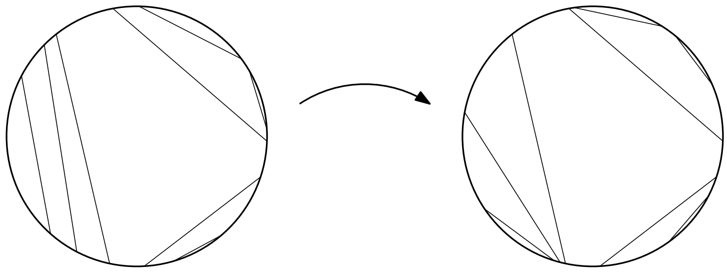

FIGURE 2. A left earthquake map.

The Zygmund regularity is essentially the infinitesimal version of quasisymmetry. For the problem discussed here, the infinitesimal version of Theorem 4.1 will be applied:

Theorem 4.2 ([5, 6])**.** *Given any function $\varphi : \mathbb{S}^1 \to \mathbb{R}$, φ is Zygmund if and only if there exists a bounded measured geodesic lamination μ such that:*

$$\varphi \frac{d}{d\theta} = \left.\frac{d}{dt}\right|_{t=0} \left(E^l_{t\mu}|_{\partial\mathbb{D}} \right) ,$$

where $E^l_{t\mu} : \mathbb{H}^2 \to \mathbb{H}^2$ is the left earthquake maps along the measured geodesic lamination $t\mu$.

That is, any Zygmund function φ corresponds to the first-order derivative on $\mathbb{S}^1$ of earthquake maps along a bounded measured geodesic lamination, up to identifying φ to a vector field on $\mathbb{S}^1$ using the standard trivialization of $T\mathbb{S}^1$.

Back to Minkowski geometry, the following lemma is a fundamental observation:

Lemma 4.1 ([1, Proposition 5.2])**.** *Let $\varphi : \mathbb{S}^1 \to \mathbb{R} \cup \{+\infty\}$ and suppose*

$$\varphi \frac{d}{d\theta} = \left.\frac{d}{dt}\right|_{t=0} \left(E^l_{t\mu}|_{\partial\mathbb{D}} \right) ,$$

for some left measured geodesic lamination μ. Then μ coincides with the pleating lamination μ_φ of the support function v_φ of $\mathrm{C}(\varphi)$.

Recall that the construction which associates a measured geodesic lamination μ_φ to the function φ was discussed in Subsection 4.1. Lemma 4.1 essentially describes the inverse procedure: starting from the measured geodesic lamination μ_φ, the function φ can be recovered as an infinitesimal earthquake, up to passing to the standard trivialization of $T\mathbb{S}^1$.

Moreover, applying Theorem 4.2 and Lemma 4.1, one concludes the first equivalence leading to the proof of Theorem 2.5, namely φ is Zygmund if and only if the pleating measured geodesic lamination μ_φ is bounded in the sense of Thurston.

The second equivalence then involves more directly arguments of Minkowski geometry, together with the regularity theory of Monge-Ampère equations. In fact, one needs to prove that the pleating lamination μ_φ is bounded if and only if the (unique) entire K-surface asymptotic to $\mathrm{C}(\varphi)$ has bounded principal curvatures. One implication follows from the following inequality, showed in [1, Proposition 5.7]:

$$||\mu_{\varphi_\Sigma}||_{\mathrm{Th}} \leq 2\sqrt{2(1+\cosh(1))}\left(\inf_{\mathbf{x}\in\Sigma}\lambda_1(\mathbf{x})\right)^{-1} , \tag{10}$$

where $\lambda_1(\mathbf{x})$ denotes the smallest principal curvature of Σ at the point $\mathbf{x}$. This in particular shows that, if the principal curvatures λ_1, λ_2 are bounded (since $\lambda_1\lambda_2 = |K|$ by Gauss' equation, they are bounded from below if and only if they are bounded from above), then the pleating lamination μ_{φ_Σ} is bounded.

For the converse implication, under the assumption that $||\mu_{\varphi_\Sigma}||_{\mathrm{Th}} < +\infty$, one supposes by contradiction that there exists a sequence $\mathbf{x}_n$ where one principal curvature goes to infinity. Then one applies isometries of $\mathbb{L}^3$ to obtain some new surfaces Σ_n (isometric images of the original surface Σ), so that the point $\mathbf{x}_n$ are sent to a compact region of $\mathbb{L}^3$. Using the assumption, one then obtains convergence of the surfaces Σ_n, and a contradiction is derived using some *a priori* estimates for Monge-Ampère equations.

5. Parameterization of spaces of entire K-surfaces

In this final section we will prove some parameterization results for the spaces of convex entire K-surfaces Σ in $\mathbb{L}^3$ which:

- Have bounded principal curvatures, or:
- Are asymptotic to a deformation of the cone of bounded type,

up to isometries of $\mathbb{L}^3$. In fact, Corollary 2.1 and Theorem 2.5 essentially completed the classification of those two types of K-surfaces, for every $K < 0$, in terms of their asymptotic functions φ_Σ. The results of this section will give a more concrete classification in terms of certain geometric parameter spaces.

We will first consider the case of entire K-surfaces with bounded principal curvatures. Let us define the space of those surfaces:

$$\mathcal{G}^K_{\text{bdd curv}} := \{\text{convex entire } K\text{-surfaces with bounded principal curvatures}\} .$$

5.1. Universal Teichmüller space.

Our main theorem will provide a parameterization of the elements of $\mathcal{G}^K_{\text{bdd curv}}$, up to translations, in terms of universal Teichmüller space. Let us briefly recall its definition. Let us denote

$$\mathcal{QS}(\mathbb{S}^1) = \{h : \mathbb{S}^1 \to \mathbb{S}^1 \text{ quasisymmetric homeomorphisms}\} .$$

There is a natural injective homomorphism

$$j : \mathrm{Isom}(\mathbb{H}^2) \to \mathcal{QS}(\mathbb{S}^1) ,$$

where, if ϕ is an isometry of $\mathbb{H}^2$, $j(\phi)$ is the trace of ϕ on the boundary, namely $j(\phi) = \phi|_{\partial\mathbb{D}}$ (in the Klein model, for instance). It turns out that $j(\mathrm{Isom}(\mathbb{H}^2))$ is a normal subgroup of $\mathcal{QS}(\mathbb{S}^1)$.

Definition 5.1. *Universal Teichmüller space is defined as the quotient:*

$$\mathcal{T}(\mathbb{D}) := \mathcal{QS}(\mathbb{S}^1)/j(\mathrm{Isom}(\mathbb{H}^2)) .$$

It turns out that $\mathcal{T}(\mathbb{D})$ is naturally endowed with an infinite-dimensional Banach manifold structure. Moreover, the tangent space at the identity of $\mathcal{T}(\mathbb{D})$ is identified to:

$$T_{[\mathrm{id}]}\mathcal{T}(\mathbb{D}) = \mathcal{Z}(\mathbb{S}^1)/j_*(\mathfrak{isom}(\mathbb{H}^2)) \ ,$$

where

$$\mathcal{Z}(\mathbb{S}^1) = \{\text{Zygmund fields on } \mathbb{S}^1\} \ ,$$

and j_* maps an element of the Lie algebra of $\mathrm{Isom}(\mathbb{H}^2)$ to the trace on $\partial\mathbb{D}$ of the corresponding Killing field on $\mathbb{H}^2$.

5.2. **Parameter space for entire K-surfaces with bounded principal curvatures.** We can now state our parameterization theorem:

Theorem 5.1. *For every $K < 0$, there is a natural bijection between the following spaces:*

- $\mathcal{G}^K_{bdd\ curv}/\mathbb{R}^3$*, where* $\mathbb{R}^3$ *acts by translations on the space of entire K-surfaces with bounded principal curvatures;*
- $T_{[\mathrm{id}]}\mathcal{T}(\mathbb{D})$*, i.e. the tangent space at the identity of universal Teichmüller space;*
- *The space of bounded measured geodesic laminations of* $\mathbb{H}^2$*.*

Proof. Theorem 2.3 and Theorem 2.5 provided a 1-1 correspondence between $\mathcal{G}^K_{\text{bdd curv}}$ and the space of Zygmund vector fields, by means of the map

$$\Sigma \in \mathcal{G}^K_{\text{bdd curv}} \mapsto \varphi_\Sigma \frac{d}{d\theta} \ .$$

Hence it remains to show that $\Sigma', \Sigma \in \mathcal{G}^K_{\text{bdd curv}}$ differ by a translation if and only if $\varphi_\Sigma(d/d\theta)$ and $\varphi_{\Sigma'}(d/d\theta)$ differ by an element in $j_*(\mathfrak{isom}(\mathbb{H}^2))$.

For this purpose, let us take an element of the Lie algebra $\mathfrak{isom}(\mathbb{H}^2)$, which is identified to a matrix in $\mathfrak{so}(2,1)$. There is a natural isomorphism $\mathbb{R}^3 \cong \mathfrak{so}(2,1)$ which associates to $v \in \mathbb{R}^3$ the linear application

$$w \mapsto v \boxtimes w \ ,$$

where $\boxtimes$ denotes the Minkowski cross product. Now, to compute Killing vector fields of $j_*(\mathfrak{isom}(\mathbb{H}^2))$, restricted to $\mathbb{S}^1$, we need to pick at every point point $(\cos\theta, \sin\theta, 1)$ which lies in $\mathbb{S}^1$ (on the plane at height $x_3 = 1$), the vector

$$(a, b, c) \boxtimes (\cos\theta, \sin\theta, 1)$$

for some fixed $a, b, c \in \mathbb{R}$, and project it down to the tangent line to the circle $\mathbb{S}^1$ at height 1. Namely, a vector field in $j_*(\mathfrak{isom}(\mathbb{H}^2))$ is of the form

$$\langle (a, b, c) \boxtimes (\cos\theta, \sin\theta, 1), (-\sin\theta, \cos\theta, 0)\rangle \frac{d}{d\theta} \ .$$

By a direct computation, we obtain that Killing vector fields all have the form

$$(-a\cos\theta - b\sin\theta + c)\frac{d}{d\theta} \ ,$$

for a, b, c arbitrary. Hence, using Lemma 2.2 (in particular Equation (9)), we see that

$$\varphi_\Sigma \frac{d}{d\theta} - \varphi_{\Sigma'} \frac{d}{d\theta} \in j_*(\mathfrak{isom}(\mathbb{H}^2)) \tag{11}$$

if and only if

$$(\varphi_\Sigma - \varphi_{\Sigma'})(e^{i\theta}) = -a\cos\theta - b\sin\theta + c\ .$$

Again by Equation (9), this is equivalent to thet fact that φ_Σ is the asymptotic function of the surface $\Sigma' + (a,b,c)$. By the uniqueness part of Theorem 2.3, we conclude that (11) is equivalent to $\Sigma = \Sigma' + (a,b,c)$, and this concludes the proof of the bijection between the first two spaces.

Finally, the bijection with the space of bounded measured geodesic laminations is induced by the map

$$\varphi \in \mathcal{Z}(\mathbb{S}^1) \mapsto \mu_\varphi\ , \tag{12}$$

where μ_φ is the pleating measured geodesic lamination introduced in Subsection 4.1. We have showed that

$$\varphi\frac{d}{d\theta} - \varphi'\frac{d}{d\theta} \in j_*(\mathfrak{isom}(\mathbb{H}^2))$$

if and only if $\varphi - \varphi'$ is the restriction on $\partial\mathbb{D}$ of an affine function on $\mathbb{R}^2$, which is also equivalent to the fact that $\mu_\varphi = \mu_{\varphi'}$. Indeed, changing φ by an affine function, also changes the convex envelope v_φ by the same affine function, and therefore leaves the pleating measured geodesic lamination invariant; the converse holds similarly. In conclusion, this shows that the map defined in (12) induces a bijection from $T_{[\mathrm{id}]}\mathcal{T}(\mathbb{D}) = \mathcal{Z}(\mathbb{S}^1)/j_*(\mathfrak{isom}(\mathbb{H}^2))$ to the space of bounded measured geodesic laminations, and thus concludes the proof. □

Remark 5.1. *In [13] it was actually proved that the bijection between $T_{[\mathrm{id}]}\mathcal{T}(\mathbb{D})$ and the space of bounded measured geodesic laminations is a homeomorphism with respect to the topology induced by the Zygmund norm on $T_{[\mathrm{id}]}\mathcal{T}(\mathbb{D})$ (which makes it a Banach space) and the uniform weak* topology for measured geodesic laminations. We wonder if this topology can be translated in terms of some type of convergence of the corresponding entire K-surfaces.*

Now, recall that the group of orientation-preserving, time-preserving isometries of $\mathbb{L}^3$ is $\mathrm{Isom}_0(\mathbb{L}^3) \cong \mathrm{SO}_0(2,1) \rtimes \mathbb{R}^3$. The subgroup $\mathbb{R}^3$ of translations is a normal subgroup, with

$$\mathrm{Isom}_0(\mathbb{L}^3)/\mathbb{R}^3 \cong \mathrm{SO}_0(2,1)\ . \tag{13}$$

Using the natural isomorphism between the Lie algebra $\mathfrak{isom}(\mathbb{H}^2) \cong \mathfrak{so}(2,1)$ and $\mathbb{R}^3$, which is equivariant for the adjoint action of $\mathrm{SO}_0(2,1)$ on $\mathfrak{so}(2,1)$ and the standard action of $\mathrm{SO}_0(2,1)$ on Minkowski space, it also turns out that there is an isomorphism $\mathrm{Isom}_0(\mathbb{L}^3) \cong \mathrm{SO}_0(2,1) \rtimes \mathfrak{so}(2,1)$ with

$$\mathrm{Isom}_0(\mathbb{L}^3)/\mathfrak{so}(2,1) \cong \mathrm{SO}_0(2,1)\ . \tag{14}$$

Hence we have the following actions of $\mathrm{SO}_0(2,1)$:

- On $\mathcal{G}^K_{\mathrm{bdd\ curv}}/\mathbb{R}^3$, induced by the action of $\mathrm{Isom}_0(\mathbb{L}^3)$ on $\mathcal{G}^K_{\mathrm{bdd\ curv}}$, using Equation (13);
- On $T_{[\mathrm{id}]}\mathcal{T}(\mathbb{D})$, using Equation (14): this essentially means that $[\varphi(d/d\theta)]$ and $[\varphi'(d/d\theta)]$ are in the same orbit if there exists $m \in \mathrm{Isom}_0(\mathbb{H}^2)$ such that $m_*(\varphi(d/d\theta)) - \varphi'(d/d\theta)$ is in $j_*(\mathfrak{isom}(\mathbb{H}^2))$;
- On the space of bounded measured geodesic laminations of $\mathbb{H}^2$, by isometries of $\mathbb{H}^2$.

By developing the definitions, using Lemma 2.2 and the above $\mathrm{SO}_0(2,1)$-equivariance of group isomorphisms, one then has the following corollary:

Corollary 5.1. *For every $K < 0$, there is a natural bijection between the following spaces:*

- $\mathcal{G}^K_{bdd\ curv}/\mathrm{Isom}_0(\mathbb{L}^3)$, *where* $\mathrm{Isom}_0(\mathbb{L}^3)$ *acts by isometries of* $\mathbb{L}^3$*;*
- *The quotient* $T_{[\mathrm{id}]}\mathcal{T}(\mathbb{D})/\mathrm{Isom}_0(\mathbb{H}^2)$*;*
- *The space of* $\mathrm{Isom}_0(\mathbb{H}^2)$*-orbits of bounded measured geodesic laminations of* $\mathbb{H}^2$.

We remark that the quotients in Corollary 5.1 are not endowed with an infinite-dimensional manifold structure, since the actions of $\mathrm{SO}_0(2,1)$ on the spaces of Theorem 5.1 are not free. In fact, the standard hyperboloid of Example 1.1, which corresponds to the zero element in the vector space $T_{[\mathrm{id}]}\mathcal{T}(\mathbb{D})$ and to the empty measured geodesic lamination, is fixed by the entire group $\mathrm{SO}_0(2,1)$. The quotient $\mathcal{G}^K_{\mathrm{bdd\ curv}}/\mathrm{Isom}_0(\mathbb{L}^3)$ will thus have a stratified singular structure. A description of such stratification is beyond the scope of this paper.

5.3. **Other parameter spaces.** By applying the same arguments as in Subsection 5.2, in an even simpler way, one can deduce from Theorem 2.3 the following parametrization result. Let us denote

$$\mathcal{G}^K_{\mathrm{bdd\ type}} := \{\text{convex entire } K\text{-surfaces asympotic to } \mathrm{C}(\varphi) \text{ of bounded type}\}\ .$$

Then we have:

Theorem 5.2. *For every $K < 0$, there is a natural bijection between the following spaces:*

- $\mathcal{G}^K_{bdd\ tyoe}/\mathbb{R}^3$, *where* $\mathbb{R}^3$ *acts by translations on the space of convex entire K-surfaces asymptotic to a deformation of the cone of bounded type;*
- *The quotient of the vector space of bounded, lower-semicontinuous functions $\varphi : \mathbb{S}^1 \to \mathbb{R}$ by the subspace $\{\varphi(e^{i\theta}) = a\cos\theta + b\sin\theta + c : a, b, c \in \mathbb{R}\}$.*

As in Corollary 5.1, one can then describe $\mathcal{G}^K_{\mathrm{bdd\ type}}/\mathrm{Isom}_0(\mathbb{L}^3)$ by a bijection with the $\mathrm{Isom}_0(\mathbb{H}^2)$-quotient of the latter vector space.

Analogously, we can reinterpret Conjecture 2.1 in this light. If we denote

$$\mathcal{G}^K := \left\{\text{convex entire } K\text{-surfaces in } \mathbb{L}^3\right\}\ ,$$

we can formulate the following:

Conjecture 5.1. *For every $K < 0$, there is a natural bijection between the following spaces:*

- $\mathcal{G}^K/\mathbb{R}^3$, *where* $\mathbb{R}^3$ *acts by translations on the space of convex entire K-surfaces;*
- *The quotient of the space of lower-semicontinuous functions $\varphi : \mathbb{S}^1 \to \mathbb{R}$, finite on at least three points, by the equivalence relation $\varphi \sim \varphi'$ if and only if $(\varphi - \varphi')(e^{i\theta}) = a\cos\theta + b\sin\theta + c$ for some $a, b, c \in \mathbb{R}$.*

Finally, we conclude by restating Problem 1.1, which is still an open question:

Problem 5.1. *Describe a parameter space for* $\mathcal{G}^K_{comp}/\mathbb{R}^3$ *or* $\mathcal{G}^K_{comp}/\mathrm{Isom}_0(\mathbb{L}^3)$, *where:*

$$\mathcal{G}^K_{comp} := \left\{ \textit{complete convex entire } K\textit{-surfaces in } \mathbb{L}^3 \right\} \ .$$

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